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LETTER TO THE EDITOR

Random walks on directed percolation clusters

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Abstract. An asymptotically exact expression is given for the mean displacement $\langle R(t)\rangle$ of random walks on directed percolation clusters on lattices in arbitrary dimensions. The critical behaviour of $R_\infty = \lim_{t\to\infty}\langle R(t)\rangle$, the mean squared displacement and the relaxation time is discussed near the threshold probability $p_c = 1$ in terms of critical exponents.

Many versions of the percolation problem have been studied since the classical work of Broadbent and Hammersley (1957). Recently more attention has been focused on developing sophisticated models which may meet theoretical interest as well as possible applications in various fields (for references see Stauffer (1979) and Essam (1980)).

In the case of directed percolation the bonds transmit only in one direction. This direction can be specified in a number of ways: the most common model is obtained when a bond is directed like a unit vector parallel to it and having positive projection on a fixed vector, playing the role of an external field. Thus on the square lattices the bonds are directed downward or to the right (see figure 1). This model is in a different universality class from ordinary percolation (Blease 1977a, b, Kertész and Vicsek 1980, Obukhov 1980, Kinzel and Yeomans 1981) and has relevance to

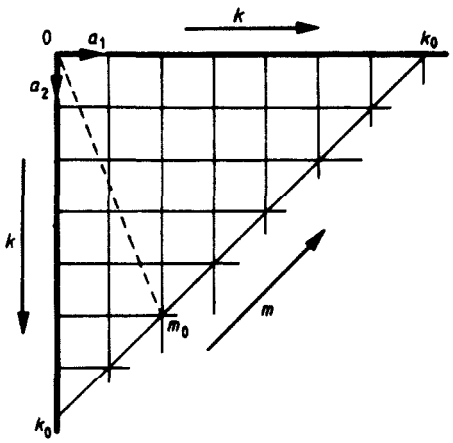

Figure 1. Part of the directed square lattice. All bonds are directed like bonds a_1 and a_2. The distance travelled by a particle starting from the origin and arriving at the point m_0 after $k = k_0$ steps is marked by a broken line.

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phenomena occurring in various systems, such as hopping conduction in a semiconductor (Kertész and Vicsek 1980, Van Lien and Shklovskii 1981), Markov process with branching, chemical processes, absorption and recombination (Grassberger and de la Torre 1979).

Random walks on ordinary percolation clusters have been investigated mainly in connection with diffusion in disordered lattices. Since the introduction of the problem called 'the ant in the labyrinth' (de Gennes 1976), a few papers have dealt with the description of the mean squared displacement $\langle R^2(t)\rangle$ of the ant as a function of time. The main quantities of the problem, the critical exponent for the long-time limit of $\langle R^2(t)\rangle$ and the associated relaxation time were obtained for site (Roussenq 1980) and bond (Vicsek 1981) percolation problems by Monte Carlo and in a closed form for the Bethe lattice (Straley 1980) and for one-dimensional bond percolation (Odagaki and Lax 1980).

A natural combination of the above models is a process defined by random walks on directed percolation clusters, which may serve as a model for the electron hopping in a strong external field, when hops opposite to the field are realised with a relatively small probability (Kertész and Vicsek 1980, Van Lien and Shklovskii 1981).

In this letter we investigate the case when no hops are allowed in the direction opposite to the external field; of course this model is an extreme version of the process occurring in a real system.

There is a correspondence between random walks on undirected percolation networks and the electrical conductivity of random resistor networks, since the diffusion constant defined from the mean squared displacement of a particle can be related to the conductivity of a resistor network (Mitescu and Roussenq 1976). This analogy, however, does not exist in the case of directed models: the conductivity of a random network of diodes cannot be obtained from the study of random walks; moreover—as will be shown later—even the critical points of the two models are different.

We define the random walk of a particle on the directed lattice in the following way. The particle starts moving from an arbitrary site along the directed bonds occupied with probability p. It is allowed to make a directed move in every unit time. The bond through which the step is made is randomly chosen from the occupied ones leading away from the given site. If there are no such bonds the particle stops moving.

Let us consider a lattice of coordination number $z = 2l$ with l directed bonds leading to and out from every site ($l = 2$ for the square, $l = 3$ for the triangular and the simple cubic lattices). Furthermore we call this an l lattice. With probability $(1-p)^l$ the particle having arrived at a site has no opportunity to proceed: it continues walking with probability $1-(1-p)^l$. The probability of completing k consecutive steps and then being blocked is $(1-p)^l[1-(1-p)^l]^k$. Therefore the mean distance of the particle from the origin after t steps (t plays the role of time) can be obtained from

$$\langle R(t)\rangle = (1-p)^l \sum_{k=0}^{t} g(k)[1-(1-p)^l]^k + g(t)[1-(1-p)^l]^{t+1} \tag{1}$$

where in the case of the square lattice

$$g(k) = k \sum_{m=0}^{k} (k!/m!(k-m)!)[1-2m/k+2m^2/k^2]^{1/2}/2^k = kf(k) \tag{2}$$

denotes the mean distance of the k-step walk. In (1) the first sum is associated with

the walks stopped not later than in the tth step, while the second term corresponds to the walks which go farther than the tth step. In deriving (2), we considered that a particle initially being at point 0 (figure 1) can reach the point m_0 in the k_0th layer in $k_0!/m_0!(k_0-m_0)!$ ways. It can be easily shown that $f(k)$ is a slowly varying function and converges for $k\to\infty$ to $1/\sqrt{2}$ in the case of the square lattice. For example $f(1)=1.0$, $f(10)=0.740$ and $f(50)=0.714$ $(1/\sqrt{2}=0.707)$, showing that the average trajectory of the particle is near the symmetry axis. Since we are interested in the long-time divergence of $\langle R(t)\rangle$, in the following we use

$$f(k)\approx F=\lim_{k\to\infty} f(k). \tag{3}$$

It can be mentioned that with a properly chosen non-Euclidean metric the factor $f(k)$ can be made unity (on the square lattice this is $|x|+|y|$ instead of $(x^2+y^2)^{1/2}$ for the distance).

Introducing $w=1-(1-p)^l$, we obtain from (1)

$$\langle R(t)\rangle=(1-w)\sum_{k=0}^{t} g(k)w^k+g(t)w^{t+1}\approx F\frac{w}{1-w}(1-w^t) \tag{4}$$

where only approximation (3) has been utilised. The simple formula (4) contains all the information we need to describe the critical behaviour of the walk. The quantities of interest are the long-time limit of $\langle R(t)\rangle$

$$R_\infty\approx Fw/(1-w) \tag{5}$$

and the relaxation time τ, defined by

$$\langle R(t)\rangle\approx R_\infty(1-a\,\mathrm{e}^{-t/\tau}). \tag{6}$$

From equations (4)–(6) we obtain

$$\tau^{-1}=\ln(1/w) \tag{7}$$

and $a=1$.

$\langle R(t)\rangle$ vanishes in isotropic percolation; thus $\langle R^2(t)\rangle$ is used there. This quantity can also be calculated for the directed model:

$$\langle R^2(t)\rangle\approx F'[w(1+w)/(1-w)^2][1-w^t-tw^{t-1}(1-w)] \tag{8}$$

with a constant F'. As far as the exponents of R and τ are concerned, (8) does not contain new information compared with (4), but the time dependence of the prefactors is changed. Next we are interested in the critical behaviour of R_∞ and τ. They diverge only when $w\to 1$; therefore the critical point of the problem is $p_c=1$, which result also shows the one-dimensional character of the directed walks. From (5)–(7) one can see that both R_∞ and τ diverge with the same exponent equal to unity if $(1-w)$ is taken as scaling parameter.

$$R_\infty\propto(1-w)^{-1}, \tag{9a}$$

$$\tau\propto(1-w)^{-1}. \tag{9b}$$

Although this result has been derived for bond percolation on l lattices of arbitrary dimensional lattices, it can be shown that (9) is universally valid for site percolation and for arbitrary lattices too, if w is properly chosen. But, if $\Delta p=1-p$ is introduced as a scaling field the exponents are no more universal as it can be demonstrated for the

case of an l lattice:

$$R_\infty \propto (\Delta p)^{-l}, \tag{10a}$$

$$\tau \propto (\Delta p)^{-l}. \tag{10b}$$

In the problem of one-dimensional, long-range percolation (Klein *et al* 1978) and the one-dimensional continuum percolation (Shalitin 1981), a similar situation occurs: universality is valid only if appropriate scaling parameters are chosen.

In conclusion, we obtained an explicit expression for the mean distance for the problem of random walks on directed lattices. The critical point of this problem is $p_c = 1$, therefore no infinite walk occurs in the system, unless all the bonds are occupied. Above the threshold of directed percolation (~ 0.64 in the square bond problem, see e.g. Kinzel and Yeomans (1981)) there are infinite directed paths, but if $p < 1$ the random walker will be caught with probability one in one of the dead branches. The directed walks belong to one universality class irrespective of the type and dimensionality of the lattice, if the scaling field is properly chosen. This universality class is characterised by the one-dimensional directed walk. (Note, the exponent of τ is different in the problem of walks on one-dimensional undirected percolation clusters (Odagaki and Lax 1980).)

An interesting generalisation of the model is the problem of random walks on a lattice in which steps against the direction of a bond are also allowed with a certain probability q. An unusual crossover is expected in this case since for any values of q different from zero the critical concentration of bonds is equal to the undirected bond percolation threshold probability of the given lattice, and the $p_c(q = 0) = 1$ value 'jumps' to the new $p_c(q \neq 0)$ value discontinuously as a function of q. Work along these lines is in progress.

The $p_c = 1$ value is also different from the critical point of a corresponding random network of diodes (Redner 1981) because a current through the diodes sets in as soon as a directed path appears in the network, but a randomly walking particle finds this path with zero probability. The diffusion–conduction analogy breaks down when nonlinear elements are introduced into the model, as expected.

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Note added. After submitting our letter a closely related letter (Stephen 1981) appeared, in which a generalised model is studied within the framework of the effective medium approximation. Part of its results disagree with ours.

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